LECTURE: 5-5 THE SUBSTITUTION RULE (PART 2)

Example 1: Evaluate the following indefinite integrals.

(a) $\int \frac{\cos\theta}{\sin^2\theta} d\theta = \int \frac{1}{u^2}\, du$

$u = \sin\theta$
$du = \cos\theta\, d\theta$

$= \int u^{-2}\, du$

$= -u^{-1} + c$

$= \boxed{\frac{-1}{\sin\theta} + c}$

(b) $\int \tan x dx = \int \frac{\sin x}{\cos x}\, dx$

$u = \cos x$
$du = -\sin x\, dx$
$\frac{du}{-\sin x} = dx$

$= \int \frac{\sin x}{u}\left(\frac{-du}{\sin x}\right)$

$= -\int \frac{1}{u}\, du$

$= \boxed{-\ln|\cos x| + c}$

$= \ln|\cos x|^{-1} + c$

$= \boxed{\ln|\sec x| + c}$

Example 2: Evaluate the following indefinite integrals.

(a) $\int (1 + \tan x)^5 \sec^2 x dx = \int u^5\, du$

$u = 1 + \tan x$
$du = \sec^2 x\, dx$

$= \frac{1}{6} u^6 + c$

$= \boxed{\frac{1}{6}(1 + \tan x)^6 + c}$

(b) $\int \frac{\cos(\pi/x)}{x^2} dx = -\frac{1}{\pi}\int \cos u\, du$

$u = \pi/x = \pi x^{-1}$
$du = -\pi x^{-2}\, dx$
$-\frac{1}{\pi}\, du = \frac{1}{x^2}\, dx$

$= -\frac{1}{\pi}\sin u + c$

$= \boxed{-\frac{1}{\pi}\sin(\pi/x) + c}$

Example 3: Evaluate $\int \frac{5 + x}{1 + x^2} dx = \int \frac{5}{1+x^2}\, dx + \underbrace{\int \frac{x}{1+x^2}\, dx}_{(**)}$

(**) $u = 1 + x^2$
$du = 2x\, dx$
$\frac{1}{2}\, du = x\, dx$

$= 5\tan^{-1} x + C_1 + \frac{1}{2}\int \frac{1}{u}\, du$

$= 5\tan^{-1} x + C_1 + \frac{1}{2}\ln|u| + C_2$

$= 5\tan^{-1} x + \frac{1}{2}\ln|1 + x^2| + c$

$= \boxed{5\tan^{-1} x + \frac{1}{2}\ln(1 + x^2) + c}$

Sometimes when you do subsitution you also end up solving for your variable in the substitution. For example:

Example 4: Evaluate $\int x^5\sqrt{x^3+1}dx = \int x^3 \cdot x^2\sqrt{x^3+1}\,dx$

$u = x^3+1$
$du = 3x^2\,dx$
$\frac{1}{3}du = x^2dx$
$u-1 = x^3$

$$= \int (u-1)\sqrt{u}\cdot\frac{1}{3}du$$
$$= \frac{1}{3}\int\left(u^{3/2} - u^{1/2}\right)du$$
$$= \frac{1}{3}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C$$
$$= \boxed{\frac{2}{15}(x^3+1)^{5/2} - \frac{2}{9}(x^3+1)^{3/2} + C}$$

Example 5: Evaluate $\int x\sqrt{x+2}dx$

$u = x+2$
$du = dx$
$x = u-2$

$$= \int (u-2)\sqrt{u}\,du$$
$$= \int\left(u^{3/2} - 2u^{1/2}\right)du$$
$$= \frac{2}{5}u^{5/2} - 2\cdot\frac{2}{3}u^{3/2} + C$$
$$= \boxed{\frac{2}{5}(x+2)^{5/2} - 4/3\,(x+2)^{3/2} + C}$$

Definite Integrals

The Substitution Rule for Definite Integrals: If g' is continuous on $[a,b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x) = \int_{g(a)}^{g(b)} f(u)du$$

input $x=b$ into your substitution to get the upper bound.

input $x=a$ into your substitution to get the lower bound.

Example 6: Evaluate $\int_0^{\pi/2}\sin^3 x\cos x dx$ two ways:

$u = \sin x$
$du = \cos x\,dx$

$x = 0$, $u = \sin 0 = 0$
$x = \pi/2$, $u = \sin\pi/2 = 1$

a) going back to x's

$$\int_0^{\pi/2}\sin^3 x\cos x\,dx = \int_{x=0}^{x=\pi/2} u^3\,du$$
$$= \frac{1}{4}u^4\Big|_{x=0}^{x=\pi/2}$$
$$= \frac{1}{4}\sin^4 x\Big|_0^{\pi/2}$$
$$= \frac{1}{4}\left((\sin\pi/2)^4 - (\sin 0)^4\right)$$
$$= \boxed{\frac{1}{4}}$$

b) using substitution

$$\int_0^{\pi/2}\sin^3 x\cos x\,dx = \int_0^1 u^3\,du$$
$$= \frac{1}{4}u^4\Big|_0^1$$
$$= \boxed{\frac{1}{4}}$$

Example 7: Evaluate the following definite integrals.

a) $\int_e^{e^3} \frac{1}{x(\ln x)^2}dx = \int_1^3 \frac{1}{u^2}\,du$

$u = \ln x$
$du = \frac{1}{x}\,dx$
$x = e,\ u = \ln e = 1$
$x = e^3,\ u = \ln e^3 = 3$

$= \int_1^3 u^{-2}\,du$

$= -u^{-1}\Big|_1^3$

$= -\frac{1}{3} + 1$

$= \boxed{2/3}$

b) $\int_1^2 x\sqrt{x-1}\,dx = \int_0^1 (u+1)\sqrt{u}\,du$

$u = x-1$
$du = dx$
$x = u+1$
$x=1 \Rightarrow u=0$
$x=2 \Rightarrow u=1$

$= \int_0^1 (u^{3/2} + u^{1/2})\,du$

$= \left(\frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right)\Big|_0^1$

$= \frac{3}{3}\frac{2}{5} + \frac{2}{3}\frac{5}{5} - 0$

$= 6/15 + 10/15$

$= \boxed{16/15}$

Example 8: Evaluate the following define integrals.

a) $\int_0^1 2^z \sin(2^z)dz = \frac{1}{\ln 2}\int_1^2 \sin u\,du$

$u = 2^z$
$du = \ln 2 \cdot 2^z$
$z=0,\ u=1$
$z=1,\ u=2$

$= -\frac{1}{\ln 2}\cos u\Big|_1^2$

$= -\frac{1}{\ln 2}(\cos(2) - \cos 1)$

$= \boxed{\frac{\cos 1 - \cos 2}{\ln 2}}$

b) $\int_0^2 \frac{x}{x^2+4}dx = \int_4^8 \frac{1}{2}\cdot\frac{1}{u}\,du$

$u = x^2+4$
$du = 2x\,dx$
$\frac{1}{2}du = x\,dx$
$x=0,\ u=4$
$x=2,\ u=8$

$= \frac{1}{2}\ln|u|\Big|_4^8$

$= \boxed{\frac{1}{2}(\ln 8 - \ln 4)}$

$= \frac{1}{2}\ln\left(\frac{8}{4}\right)$

$= \boxed{\frac{1}{2}\ln(2)}$

$= \boxed{\ln\sqrt{2}}$

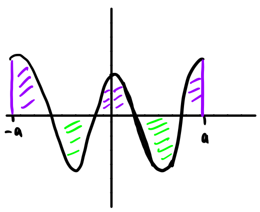
Symmetry

- A function f is even if $f(-a)=f(a)$. Even functions are symmetric about the y-axis.
- A function f is odd if $f(-a)=-f(a)$. Odd functions are symmetric about the origin.

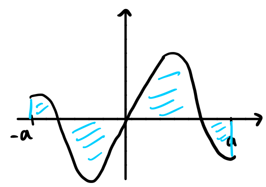
Integrals of Even/Odd Functions: Suppose a function $f(x)$ is (blank) on $[-a, a]$. Then,

(a) (even) $\int_{-a}^{a} f(x)dx = \boxed{2\int_0^a f(x)\,dx}$

often it's easier to input 0 into the anti-derivative.



(b) (odd) $\int_{-a}^{a} f(x)dx = \boxed{0}$



Example 9: Evaluate the following definite integrals.

(a) $\int_{-2}^{2}(x^2+1)dx = 2\int_0^2 (x^2+1)\,dx$

$f(x)=x^2+1$ is even!

$= 2\left(\frac{1}{3}x^3+x\right)\Big|_0^2$

$= 2\left(\frac{8}{3}+2\right) - 2\cdot 0$

$= 2\left(\frac{8}{3}+\frac{6}{3}\right)$

$= \boxed{\frac{28}{3}}$

(b) $\int_{-1}^{1}\frac{\tan x}{1+x^2}dx = \boxed{0}$

$f(x)=\frac{\tan x}{1+x^2}$

$f(-x)=\frac{\tan(-x)}{1+(-x)^2}$

$= -\frac{\tan x}{1+x^2}$

so $f(x)$ is odd!

Example 10: If f is continuous and $\int_0^9 f(x)dx = 4$, find $\int_0^3 xf(x^2)dx = \int_0^9 \frac{1}{2} f(u)\,du$

$u=x^2$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$x=0,\ u=0^2=0$

$x=3,\ u=3^2=9$

$= \frac{1}{2}\int_0^9 f(u)\,du$

$= \frac{1}{2}(4)$

$= \boxed{2}$